A GENERALIZATION OF EULER'S CONSTANT

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1. History of γ and the intent to study $\sum_{k=0}^{n} \frac{1}{\alpha k+1}$.

The Euler–Mascheroni constant, sometimes shortened to Euler's constant, is a constant in mathematics recurring in analysis and number theory and is usually denoted by the lowercase Greek letter γ (gamma). It is defined as the limiting difference between the harmonic series and the natural logarithm:

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k+1} - \ln(n+1) \right).$$

Our intent in this paper is to examine how the above equation changes, and how the constant changes, when we consider the more general expression

$$\begin{aligned} \delta(\alpha) &= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{1}{\alpha k+1} - \int_0^n \frac{1}{\alpha x+1} dx \right) \\ &= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{1}{\alpha k+1} - \frac{1}{\alpha} \cdot \ln(\alpha n+1) \right), \end{aligned}$$

for various values of $\alpha > 0$. Note that $\delta(1) = \gamma$.

Leonhard Euler was born on April 15, 1707 in Basel Switzerland. His father, Paul Euler, a pastor of the Reformed Church, and his mother, Marguerite Brucker, the daughter of a pastor, had three children: Leonhard, Anna Maria and Maria Magdalena. Soon after the birth of their first son, Leonhard, the Eulers moved from Basel to the town of Riehen, where they would remain for the remainder of his early childhood. Johann Bernoulli, a family friend, who was then regarded as Europe's foremost mathematician, would prove to be a great influence on Leonhard. Euler was sent back to Basel to live with his maternal grandmother and begin his formal education. When Leonhard became thirteen he enrolled at the University of Basel, and in 1723, received his Master of Philosophy with a dissertation that compared the philosophies of Descartes and Newton. During this time, Leonhard received Saturday afternoon tutorials from Johann Bernoulli, who saw fit to nurture the incredible talent for mathematics that his pupil displayed. Conversely, Paul Euler was encouraging the study of theology, Greek, and Hebrew in the hope that his son was to become a pastor; however, Bernoulli was able to convince Paul Euler that Leonhard was destined to become a great mathematician.

In 1726, Euler completed a dissertation on the propagation of sound entitled "*De Sono*." Around this time Johann Bernoulli's two sons, Daniel and Nicolas, were working at the Imperial Russian Academy of Sciences in St Petersburg. On July 10, 1726, Nicolas died of appendicitis after spending a year in Russia. Daniel assumed his brother's position in the mathematics/physics division; he recommended to the Academy that the post in physiology that he had vacated should be filled by his friend Euler. In November 1726 Euler accepted the offer, but delayed making the trip to St Petersburg while he unsuccessfully applied for a physics professorship at the University of Basel.

Euler arrived in the Russian capital, St Petersburg, on 17 May 1727. He was promoted from his junior post in the physiology department of the academy to a position in the mathematics department. He lived with Daniel Bernoulli with whom he often worked in close collaboration. Euler mastered Russian and settled into life in St Petersburg.

Peter the Great established The Academy at St Petersburg. The Academy was intended to improve education in Russia and to close the scientific gap that existed with Western Europe. As a result, The Academy was made especially attractive to foreign scholars like Euler. The Academy possessed plentiful

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financial resources and a comprehensive library culled from the private libraries of Peter himself and other nobility. There were very few students enrolled in the Academy; the purpose of which to lessen the faculty's teaching burden, and emphasize research and offer its faculty both the time and the freedom to pursue science.

The Academy's benefactress, Catherine I, who had continued the progressive policies of her late husband, died on the day of Euler's arrival. The Russian nobility then gained power upon the ascension of the twelve-year-old Peter II. The nobility were suspicious of the Academy's foreign scientists, and thus cut funding and caused other difficulties for Euler and his colleagues; however, upon the death of Peter II, the conditions at the Academy improved slightly and Euler swiftly rose through the ranks and was made professor of physics in 1731. Two years later, fed up with the censorship and hostility he faced, Daniel Bernoulli, left St Petersburg and Euler succeeded him as the head of the mathematics department.

Euler married Katharina Gsell (1707–1773), a daughter of Georg Gsell, a painter from the Academy Gymnasium on January 7, 1734. They bought a house on the Neva River and of their thirteen children only five survived childhood.

In a 1735 paper titled "*De Progressionibus harmonicis observationes*" (Eneström Index 43), Euler debuted his constant γ, for which he used the notations C and O. *De Progressionibus harmonicis observationes* reflected Euler's interest in zeta functions which led to the existence of γ. Julian Havil, in his book "*Gamma*," walks us through the birth of γ. Using the result

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots, \qquad -1 < x \le 1,$$

he replaced x with $\frac{1}{r}$ to get

$$\ln\left(1+\frac{1}{r}\right) = \frac{1}{r} - \frac{1}{2r^2} + \frac{1}{3r^3} - \frac{1}{4r^4} + \cdots,$$

whence

$$\frac{1}{r} = \ln\left(\frac{r+1}{r}\right) + \frac{1}{2r^2} - \frac{1}{3r^3} + \frac{1}{4r^4} - \cdots.$$

Summing this expression over n, yields

$$\sum_{r=1}^{n}\frac{1}{r} = \sum_{r=1}^{n}\ln\left(\frac{r+1}{r}\right) + \frac{1}{2}\sum_{r=1}^{n}\frac{1}{r^2} - \frac{1}{3}\sum_{r=1}^{n}\frac{1}{r^3} + \frac{1}{4}\sum_{r=1}^{n}\frac{1}{r^4} - \cdots.$$

Therefore,

$$\sum_{r=1}^{n}\frac{1}{r} = \sum_{r=1}^{n}\left(\ln(r+1) - \ln(r)\right) + \frac{1}{2}\sum_{r=1}^{n}\frac{1}{r^2} - \frac{1}{3}\sum_{r=1}^{n}\frac{1}{r^3} + \frac{1}{4}\sum_{r=1}^{n}\frac{1}{r^4} - \cdots,$$

and

$$\sum_{r=1}^{n}\frac{1}{r} = \ln(n+1) + \frac{1}{2}\sum_{r=1}^{n}\frac{1}{r^2} - \frac{1}{3}\sum_{r=1}^{n}\frac{1}{r^3} + \frac{1}{4}\sum_{r=1}^{n}\frac{1}{r^4} - \cdots,$$

which makes

$$\sum_{r=1}^{n}\frac{1}{r} - \ln(n+1) = \frac{1}{2}\sum_{r=1}^{n}\frac{1}{r^2} - \frac{1}{3}\sum_{r=1}^{n}\frac{1}{r^3} + \frac{1}{4}\sum_{r=1}^{n}\frac{1}{r^4} - \cdots.$$

In the limit as $n \to \infty$, we have the difference between the divergent harmonic series and the divergent natural logarithm expressed in terms of an infinite number of convergent zeta series, the sum of which would therefore be very nice to know.

Definition 1.

$$\zeta(k) = \sum_{r=1}^{\infty}\frac{1}{r^k}.$$

Euler established that

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6},$$

and more generally how to find $\zeta(k)$ for k even[7]. Now, with the known values for $\zeta(2), \zeta(4), \cdots, \zeta(26)$, and the ability to approximate $\zeta(k)$ for k odd, since no known formula exists for these values, Euler is able to produce γ to five decimal places of accuracy using perhaps his most elegant representation of gamma:

$$\gamma = \frac{1}{2}\zeta(2) - \frac{1}{3}\zeta(3) + \frac{1}{4}\zeta(4) - \cdots .$$

In subsequent papers Euler was able to increase his accuracy to 12, and later, 16 decimal places of accuracy. Euler regarded γ as being worthy of serious consideration. Years later, in 1781, Euler composed *De Numero Memorabili in Summatione Progressionis Harmonicae Naturalis Occurente,* which was devoted entirely to the study of γ. Euler hoped that γ was the logarithm of some other number of importance. He was unable to identify any such number; however he was able to approximate γ using lists of series.

The continuing political turmoil in Russia concerned Euler and on the 19th of June 1741 Euler departed St Petersburg to take up a post at the Berlin Academy at the request of Frederick the Great of Prussia. Euler was made a member of the Academy of Sciences and a professor of mathematics. Euler's time at the Berlin Academy proved to be very fruitful and in his twenty-five years at the Berlin Academy, Euler wrote over 380 articles and composed the two works for which he would be most renowned: the *Introductio in analysin infinitorum*, a text on functions published in 1748, and the *Institutiones calculi differentialis*, published in 1755 on differential calculus. At the same time he continued his philosophical contributions to the Academy of St Petersburg, which granted him a pension in 1742. In 1755, he was elected a foreign member of the Royal Swedish Academy of Sciences.

Despite Euler's immense contribution to the prestige of the Berlin Academy, he was forced to leave Berlin. The need for his departure was, in part, due to a conflict of personality between himself and Frederick the Great. Frederick came to regard Euler as unsophisticated in comparison to the circle of philosophers the German king brought to the Berlin Academy. Euler, a simple religious man and a hard worker, was very conventional in his beliefs and tastes and had limited training in rhetoric. Euler tended to debate matters that he knew little about, making him a frequent target of Frederick the Great's contemporaries which included Voltaire who was famous for his wit and for his advocacy of civil liberties, freedom of religion and free trade. Frederick mockingly referred to Euler as "Cyclops," a reference to the blindness that Euler had suffered in his right eye as a result of a near-fatal fever Euler suffered in 1735.

In 1766, Euler obtained permission from the King of Prussia to return to St Petersburg. Soon after his return to the Imperial Russian Academy a cataract formed in his left eye, which ultimately deprived him almost entirely of sight. Euler continued to publish works, despite his blindness, by dictating to his servant, a tailor's apprentice who was absolutely devoid of any mathematical knowledge. Despite this handicap, Euler's productivity actually increased. He produced on average one mathematical paper every week in the year 1775.

Another task to which he set himself was the preparation of his *Lettres à une princesse d'Allemagne sur quelques sujets de physique et de philosophie* (Letters of Euler on different subjects in natural philosophy addressed to a German Princess). They were written at the request of the princess of Anhalt-Dessau. These writings contain an exposition of the principal facts of mechanics, optics, acoustics and physical astronomy as well as offering insights into Euler's personality and religious beliefs. This book became more widely read than any of his mathematical works.

Euler's second stay in Russia was marred by tragedy. A fire in St Petersburg in 1771 cost him his home, the destruction of the greater part of his property and almost his life. He was only saved by the courage of a native of Basel, Peter Grimmon, who carried him out of the burning house.

In 1773, he lost his wife Katharina after 40 years of marriage. Three years after his wife's death Euler married her half sister, Salome Abigail Gsell (1723–1794). This marriage would last until his death.

On the 18th of September 1783, while he was amusing himself at tea with one of his grandchildren, Anders Lexell, about the newly discovered Uranus and its orbit, Euler died of apoplexy. He was buried next to Katharina, his first wife, at the Smolensk Lutheran Cemetery on Vasilievsky Island. In 1837, Russian Academy of Sciences put a headstone on his grave, which in 1956, for the 250th anniversary of Euler's birth, was moved together with his remains to the necropolis of 18th century at Alexander Nevsky Lavra.

Our plan is to generalize the discussion of Euler's Constant in the book "*Gamma*", by Julian Havil. In that book, γ is described as the limit, as n goes to infinity, of the difference between the sum of discrete rectangles, whose area can be represented as: $\sum\limits_{k=0}^{n} \frac{1}{k+1}$, and the area under the curve, from zero to n, of the function: $f(x) = \frac{1}{x+1}$. Now, for $\delta > 0$, we define $\delta(\alpha)$ to be the limit, as n goes to infinity, of the difference between the sum of the area of the rectangles, $\sum\limits_{k=0}^{n} \frac{1}{\alpha k+1}$, and the area under the curve, from zero to n, of the function: $f(x) = \frac{1}{\alpha x+1}$. Since δ is dependent on the choice of α, we can write it as a function of α:

$$\delta(\alpha) = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{1}{\alpha k+1} - \int_{0}^{n} \frac{1}{\alpha x+1} dx \right)$$

There are in fact two different ways to look at $\delta(\alpha)$ as a limit. If we are going to apply the Euler-MacLaurin summation formula, as in section 5 of this paper, then we should indeed consider $\delta(\alpha) = \lim_{n\to\infty} \widehat{\delta}_n(\alpha)$ where

$$\begin{aligned} \widehat{\delta}_n(\alpha) &= \sum_{k=0}^{n} \frac{1}{\alpha k+1} - \int_{0}^{n} \frac{1}{\alpha x+1} dx \\ &= \sum_{k=0}^{n} \frac{1}{\alpha k+1} - \frac{1}{\alpha} \cdot \ln(\alpha n+1). \end{aligned}$$

However, a more basic approach is to regard $\sum\limits_{k=0}^{n} \frac{1}{\alpha k+1}$ as an upper sum for the integral $\int_0^{n+1} \frac{1}{\alpha x+1} dx$. From this viewpoint, we should consider

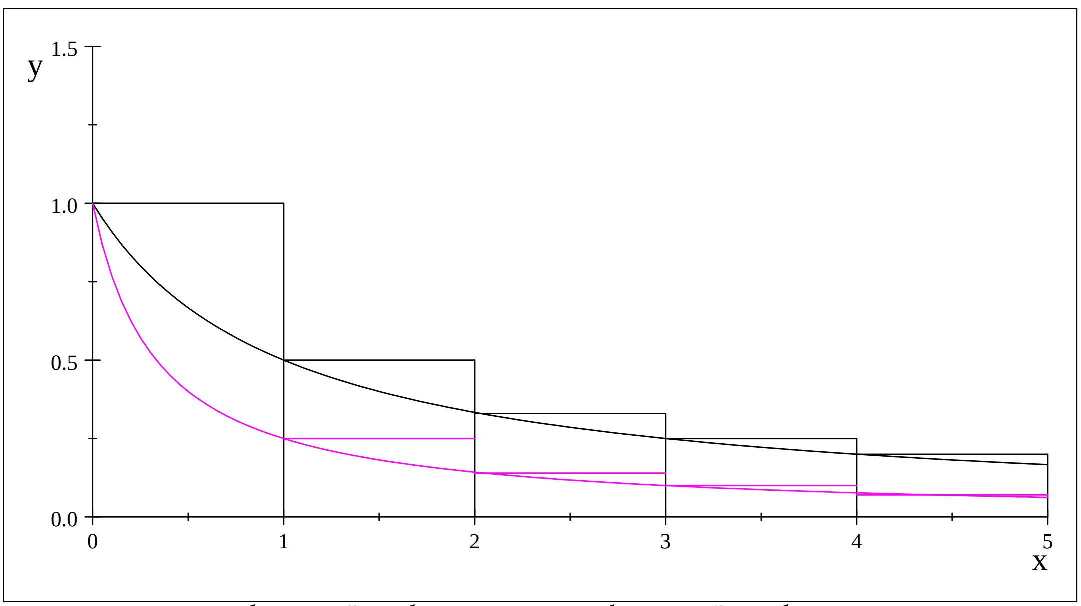
$$\begin{aligned} \delta_n(\alpha) &= \sum_{k=0}^{n} \frac{1}{\alpha k+1} - \int_{0}^{n+1} \frac{1}{\alpha x+1} dx \\ &= \sum_{k=0}^{n} \frac{1}{\alpha k+1} - \frac{1}{\alpha} \cdot \ln(\alpha(n+1)+1). \end{aligned}$$

Since

$$\begin{aligned} \widehat{\delta}_n(\alpha) - \delta_n(\alpha) &= \int_{n}^{n+1} \frac{1}{\alpha x+1} dx \\ &= \frac{1}{\alpha} \cdot \ln\left(1 + \frac{\alpha}{\alpha n+1}\right), \end{aligned}$$

which goes to zero as n goes to infinity, both expressions have the same limit:

$$\delta(\alpha) = \lim_{n\to\infty} \widehat{\delta}_n(\alpha) = \lim_{n\to\infty} \delta_n(\alpha).$$



$f(x) = \frac{1}{x+1}$ vs $\sum_{k=0}^{n} \frac{1}{k+1}$ and $f(x) = \frac{1}{\alpha x+1}$ vs $\sum_{k=0}^{n} \frac{1}{\alpha k+1}$ (in pink)

The integral $I_n = \int_0^{n+1} \frac{1}{\alpha x+1} dx$, has an upper sum $U_n = \sum_{k=0}^{n} \frac{1}{\alpha k+1}$, and a corresponding lower sum $L_n = \sum_{k=1}^{n+1} \frac{1}{\alpha k+1}$. The upper and lower sums differ only in the first and last terms, 1 and $\frac{1}{\alpha(n+1)+1}$. Moreover, note that the following hold:

$$L_n < I_n < U_n \text{ for all } n > 0.$$

$$\delta_n = U_n - I_n < U_n - L_n = 1 - \frac{1}{\alpha\,(n+1)+1} < 1.$$

Meanwhile,

$$\delta_1(\alpha) \leq \delta_2(\alpha) \leq \delta_3(\alpha) \cdots$$

Therefore, the sequence is an increasing sequence bounded by 1, so that limit $\delta(\alpha) = \lim_{n\to\infty} \delta_n(\alpha)$ exists. In fact, as α gets large, $\delta(\alpha)$ approaches 1, the least upper bound of $\delta(\alpha)$. Observe that

$$1 > \delta(\alpha) > 1 - \int_0^1 \frac{1}{\alpha x+1} dx,$$

and

$$1 - \int_0^1 \frac{1}{\alpha x+1} dx = 1 - \frac{1}{\alpha} \ln(\alpha+1).$$

We note that

$$\lim_{\alpha\to\infty} 1 - \frac{1}{\alpha} \ln(\alpha+1) = 1,$$

and conclude that

$$\lim_{\alpha\to\infty} \delta(\alpha) = 1$$

Further, we observe that

$$\gamma = \delta(1) \quad \textit{and} \quad \gamma_n = \delta_n(1).$$

2. Formulas for $\delta(\alpha)$ and $\delta_n(\alpha)$.

Now we will find formulas for $\delta(\alpha)$ and $\delta_n(\alpha)$. After the formulas have been established, we will use those to make tables of $\delta(\alpha)$.

Theorem 2. *If $\alpha > 0$, then*

$$\delta_n(\alpha) = \frac{1}{\alpha}\left(\gamma_n - \ln(\alpha) + (\alpha-1)\sum_{k=0}^{n} \frac{1}{(k+1)(\alpha k+1)} + \varepsilon_n\right), \quad \textit{where } \varepsilon_n = \ln\left(\frac{n+2}{n+1+\frac{1}{\alpha}}\right). \tag{1}$$

$$\delta(\alpha) = \frac{1}{\alpha}\left(\gamma - \ln\alpha + (\alpha-1)\sum_{k=0}^{\infty} \frac{1}{(k+1)(\alpha k+1)}\right). \tag{2}$$

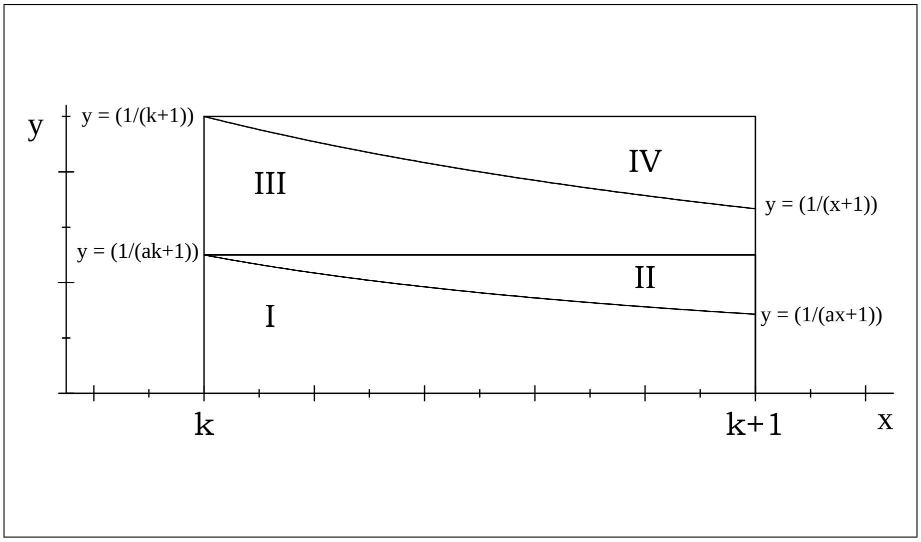
Note that the sums, in both (1) and (2), converge; thus, this is an effective was to find $\delta(\alpha)$. There is no explicit formula for the sums, however, their convergence is quick enough that it may be calculated accurately on a computer.

Now we are going to analyze the area between the curves $\frac{1}{x+1}$, $\frac{1}{\alpha x+1}$, and the corresponding rectangles between $x = k$ and $x = k+1$.

Notation 3. *We omit α in $\delta(\alpha)$ for this calculation as it is fixed. Let*

$$\Delta\delta = \delta_k - \delta_{k-1}$$
$$\Delta\gamma = \gamma_k - \gamma_{k-1}$$

Now let's look at an arbitrary unit of the graph on page 4. This figure, on page 6, is drawn for the case $\alpha > 1$. If $\alpha < 1$, the figure is different but the calculation is the same.



$$
\begin{aligned}
\text{Area I} &= \int_k^{k+1} \frac{1}{\alpha x+1} dx \\
&= \frac{1}{\alpha} \ln(\alpha x+1)\Big|_k^{k+1} \\
&= \frac{1}{\alpha} \ln(\alpha(k+1)+1) - \frac{1}{\alpha} \ln(\alpha k+1); \\
\text{Area I} + \text{Area II} &= \frac{1}{\alpha k+1}; \\
\text{Area II} &= \Delta\delta;
\end{aligned}
$$

$$
Note: \sum_{k=0}^{n} \frac{1}{\alpha k+1} = \frac{1}{\alpha} \ln(\alpha n+1) + \widehat{\delta}_n = \int_0^n \frac{1}{\alpha x+1} dx + \widehat{\delta}_n;
$$

$$
\begin{aligned}
\text{Area I+Area II+Area III} &= \int_k^{k+1} \frac{1}{x+1} dx \\
&= \ln(x+1)\Big|_k^{k+1} \\
&= \ln(k+2) - \ln(k+1);
\end{aligned}
$$

$$
Note: \sum_{k=0}^{n} \frac{1}{k+1} = \ln(n+1) + \widehat{\gamma}_n = \int_0^n \frac{1}{x+1} dx + \widehat{\gamma}_n;
$$

$$
\text{Area IV} = \Delta\gamma;
$$

$$
\text{Area I+Area II+Area III+Area IV} = \frac{1}{k+1};
$$

$$\begin{aligned}\text{Area III+Area IV} &= \frac{1}{k+1} - \frac{1}{\alpha k+1}\\ &= \frac{\alpha k+1-(k+1)}{(k+1)(\alpha k+1)}\\ &= \frac{(\alpha-1)k}{(k+1)(\alpha k+1)}\\ &= \frac{(\alpha-1)(k+1)}{(k+1)(\alpha k+1)} - \frac{(\alpha-1)1}{(k+1)(\alpha k+1)}\\ &= (\alpha-1)\frac{1}{(\alpha k+1)} - (\alpha-1)\frac{1}{(k+1)(\alpha k+1)}.\end{aligned}$$

The different expressions for the whole rectangle yield,

$$\begin{aligned}\text{Area I +Area II +Area III + Area IV} &= \frac{1}{k+1}\\ &= \ln(k+2) - \ln(k+1) + \Delta\gamma\\ = \frac{1}{\alpha}\ln(\alpha(k+1)+1) - \frac{1}{\alpha}\ln(\alpha k+1) + \Delta\delta &+ (\alpha-1)\frac{1}{(\alpha k+1)} - (\alpha-1)\frac{1}{(k+1)(\alpha k+1)}.\end{aligned}$$

Now we sum the rectangles:

$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{k+1} &= \ln(n+2) + \gamma_n\\ &= \frac{1}{\alpha}\ln(\alpha(n+1)+1) + \delta_n + (\alpha-1)\sum_{k=0}^{n}\frac{1}{(\alpha k+1)} - (\alpha-1)\sum_{k=0}^{n}\frac{1}{(k+1)(\alpha k+1)}\\ &= \frac{1}{\alpha}\ln(\alpha(n+1)+1) + \delta_n + (\alpha-1)\left[\frac{1}{\alpha}\ln(\alpha(n+1)+1) + \delta_n\right] - (\alpha-1)\sum_{k=0}^{n}\frac{1}{(k+1)(\alpha k+1)}\\ &= \ln(\alpha(n+1)+1) + \alpha\delta_n - (\alpha-1)\sum_{k=0}^{n}\frac{1}{(k+1)(\alpha k+1)}.\end{aligned}$$

Thus,

$$\ln(n+2) + \gamma_n = \ln(\alpha) + \ln\left(n+1+\frac{1}{\alpha}\right) + \alpha\delta_n - (\alpha-1)\sum_{k=0}^{n}\frac{1}{(k+1)(\alpha k+1)}.$$

Solving for $\alpha\delta_n$, yields

$$\alpha\delta_n = \gamma_n - \ln(\alpha) + (\alpha-1)\sum_{k=0}^{n}\frac{1}{(k+1)(\alpha k+1)} + \varepsilon_n, \text{ where } \varepsilon_n = \ln\left(\frac{n+2}{n+1+\frac{1}{\alpha}}\right),$$

whence

$$\delta_n = \frac{1}{\alpha}\left(\gamma_n - \ln(\alpha) + (\alpha-1)\sum_{k=0}^{n}\frac{1}{(k+1)(\alpha k+1)} + \varepsilon_n\right).$$

In general,

$$\sum_{k=0}^{n}\frac{1}{\alpha k+1} = \frac{1}{\alpha}\left(\ln(\alpha(n+1)+1) + \gamma_n - \ln(\alpha) + (\alpha-1)\sum_{k=0}^{n}\frac{1}{(k+1)(\alpha k+1)} + \varepsilon_n\right).$$

Taking the limit as n goes to infinity, the last term clearly goes to zero, and we obtain

$$\delta(\alpha) = \frac{1}{\alpha}\left(\gamma - \ln\alpha + (\alpha - 1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(\alpha k+1)}\right).$$

The formula of the theorem for $\delta(\alpha)$ is easily programmed. The values of $\delta(\alpha)$ for various alpha are given in the following table.

α	$\delta(\alpha)$
00.10	0.508338
00.20	0.516604
00.30	0.524785
00.40	0.532836
00.50	0.540726
00.60	0.548434
00.70	0.555944
00.80	0.563246
00.90	0.570338
01.00	0.577216
01.10	0.583882
01.20	0.590340
01.30	0.596594
01.40	0.602649
01.50	0.608513
01.60	0.614191
01.70	0.619690
01.80	0.625017
01.90	0.630178
02.00	0.635181
02.20	0.644737
02.40	0.653734
02.60	0.662218
02.80	0.670229
03.00	0.677807
03.20	0.684986
03.40	0.691797
03.60	0.698267
03.80	0.704424
04.00	0.710290
05.00	0.735920
06.00	0.756728
07.00	0.774010
08.00	0.788631
09.00	0.801191
10.00	0.812117
15.00	0.850970
20.00	0.875106
25.00	0.891776
30.00	0.904083
35.00	0.913595

We see that as alpha approaches zero, $\delta(\alpha)$ approaches $\frac{1}{2}$, while as alpha grows without bound, $\delta(\alpha)$ approaches 1. Moreover, $\delta_n \to \delta$ uniformly on $[A, \infty)$, where $0 < A$. Likewise, $\widehat{\delta}_n \to \delta$ uniformly on $[A, \infty)$, for $A > 0$. Moreover that,

$$\begin{aligned}\widehat{\delta}_n &= \frac{1}{\alpha}\left(\gamma_n - \ln\alpha + (\alpha-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(\alpha k+1)} + \ln\left(\frac{n+2}{n+1+\frac{1}{\alpha}}\right) + \ln\left(\frac{\alpha n+\alpha+1}{\alpha n+1}\right)\right)\\ &= \frac{1}{\alpha}\left(\gamma_n - \ln\alpha + (\alpha-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(\alpha k+1)} + \ln\left(\frac{n+2}{n+\frac{1}{\alpha}}\right)\right).\end{aligned}$$

Each of these terms approaches the limit

$$\frac{\gamma}{\alpha} - \frac{\ln\alpha}{\alpha} + \frac{\alpha-1}{\alpha}\sum_{k=0}^{\infty}\frac{1}{(k+1)(\alpha k+1)} + 0;$$

uniformly on $[A,\infty)$. The term $\frac{\ln\alpha}{\alpha}$ is independent of n. For n sufficiently large,

$$\frac{1}{\alpha}(\gamma-\gamma_n) \le \frac{1}{A}(\gamma-\gamma_n) < \frac{\varepsilon}{3};$$

$$\left|\frac{\alpha-1}{\alpha}\sum_{k=0}^{\infty}\frac{1}{(k+1)(\alpha k+1)}\right| \le \max_{[A,\infty)}\left|1-\frac{1}{\alpha}\right|\sum_{k=0}^{\infty}\frac{1}{(k+1)(Ak+1)} < \frac{\varepsilon}{3},$$

$$\frac{1}{\alpha}\ln\left(\frac{n+2}{n+\frac{1}{\alpha}}\right) \le \frac{1}{A}\ln\left(\frac{n+2}{n+\frac{1}{\alpha}}\right) < \frac{\varepsilon}{3}.$$

Now, continuing this line of thought, we calculate the derivative of δ with respect to α. We will need the following standard result found in Rudin's text[5].

Theorem 4. *Suppose $\{f_n\}$ a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad (a \le x \le b).$$

Starting with

$$\widehat{\delta}_n = \sum_{k=0}^{n}\frac{1}{(\alpha k+1)} - \frac{1}{\alpha}\ln(\alpha n+1),$$

we get

$$\begin{aligned}\frac{d\widehat{\delta}_n}{d\alpha} &= \sum_{k=0}^{n}\frac{-k}{(\alpha k+1)^2} + \frac{1}{\alpha^2}\ln(\alpha n+1) - \frac{n}{\alpha(\alpha n+1)}\\ &= \frac{1}{\alpha}\sum_{k=0}^{n}\frac{-\alpha k-1}{(\alpha k+1)^2} + \frac{1}{\alpha}\sum_{k=0}^{n}\frac{1}{(\alpha k+1)^2} + \frac{1}{\alpha^2}\ln(\alpha n+1) - \frac{n}{\alpha(\alpha n+1)}\\ &= -\frac{1}{\alpha}\left(\sum_{k=0}^{n}\frac{1}{(\alpha k+1)} - \frac{1}{\alpha}\ln(\alpha n+1)\right) + \frac{1}{\alpha}\sum_{k=0}^{n}\frac{1}{(\alpha k+1)^2} - \frac{1}{\alpha\left(\alpha+\frac{1}{n}\right)}\\ &= \frac{1}{\alpha}\cdot\widehat{\delta}_n + \frac{1}{\alpha}\cdot\sum_{k=0}^{n}\frac{1}{(\alpha k+1)^2} - \frac{1}{\alpha}\cdot\frac{1}{\left(\alpha+\frac{1}{n}\right)}.\end{aligned}$$

From before, $\delta_n \to \delta$ uniformly. We know $\frac{1}{\alpha}\sum_{k=0}^{n}\frac{1}{(\alpha k+1)^2}$ converges uniformly on $[A,\infty)$. Thus, by Theorem 4,

$$\delta' = \lim_{n\to\infty}\delta_n',$$

which is

$$\frac{d\delta}{d\alpha} = \frac{1}{\alpha}\left(\sum_{k=0}^{\infty}\frac{1}{(\alpha k+1)^2} - \delta - \frac{1}{\alpha}\right).$$

We can evaluate $\frac{d\delta}{d\alpha}$;

α	$\sum_k \frac{1}{(\alpha k+1)^2}$	$\frac{1}{\alpha}$	δ	$\left(\sum_k \frac{1}{(\alpha k+1)^2} - \frac{1}{\alpha} - \delta\right)$	$\frac{d\delta}{d\alpha}$
0.1000	10.5166	10.0000	0.5083	0.0082	0.0825
0.2000	5.5331	5.0000	0.5166	0.0165	0.0823
0.3000	3.8825	3.3333	0.5248	0.0244	0.0812
0.4000	3.0647	2.5000	0.5328	0.0319	0.0797
0.5000	2.5797	2.0000	0.5407	0.0390	0.0780
0.6000	2.2608	1.6667	0.5484	0.0457	0.0761
0.7000	2.0364	1.4286	0.5559	0.0519	0.0741
0.8000	1.8708	1.2500	0.5632	0.0576	0.0720
0.9000	1.7443	1.1111	0.5703	0.0629	0.0698
1.0000	1.6449	1.0000	0.5772	0.0677	0.0677
2.0000	1.2337	0.5000	0.6352	0.0985	0.0493
3.0000	1.1217	0.3333	0.6778	0.1106	0.0369
4.0000	1.0748	0.2500	0.7103	0.1145	0.0286
5.0000	1.0507	0.2000	0.7359	0.1148	0.0230
6.0000	1.0366	0.1667	0.7567	0.1132	0.0189
7.0000	1.0277	0.1429	0.7740	0.1108	0.0158
8.0000	1.0217	0.1250	0.7886	0.1081	0.0135
9.0000	1.0174	0.1111	0.8012	0.1051	0.0117
10.0000	1.0143	0.1000	0.8121	0.1022	0.0102

Numerically, we see that $\frac{d\delta}{d\alpha} > 0$ for all $\alpha > 0$, and δ is increasing and concave down. However, we have not been able to provide a proof. Also note that

$$\lim_{\alpha \to 0^+} \delta' = \frac{1}{12}.$$

This is a consequence of our Euler-MacLaurin summation formula in Section 5.

Also, on page 58 of Havil's *"Gamma,"* Julian Havil derives the equation:

$$\begin{aligned} \frac{\Gamma'(x)}{\Gamma(x)} &= -\frac{1}{x} - \gamma + \sum_{r=1}^{\infty} \left(\frac{1}{r} - \frac{1/r}{1 + x/r} \right) \\ &= -\frac{1}{x} - \gamma + \sum_{r=1}^{\infty} \left(\frac{1}{r} - \frac{1}{r+x} \right). \end{aligned}$$

Where Γ represents the gamma function.

Now, we are able to calculate

$$\begin{aligned} -\frac{1}{\alpha}\frac{\Gamma'\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)} + \frac{1}{\alpha}\ln\left(\frac{1}{\alpha}\right) &= -\frac{1}{\alpha}\left[-\alpha - \gamma + \sum_{r=1}^{\infty}\left(\frac{1}{r} - \frac{1}{r + \frac{1}{\alpha}} \right) \right] - \frac{1}{\alpha}\ln\left(\frac{1}{\alpha}\right) \\ &= -\frac{1}{\alpha}\left[-\alpha - \gamma + \sum_{k=0}^{\infty} \frac{1}{k+1} - \sum_{k=1}^{\infty} \frac{\alpha}{\alpha k + 1} \right] - \frac{1}{\alpha}\ln\left(\frac{1}{\alpha}\right) \\ &= -\frac{1}{\alpha}\left[-\gamma + \sum_{k=0}^{\infty} \frac{1}{k+1} - \sum_{k=0}^{\infty} \frac{\alpha}{\alpha k + 1} \right] - \frac{1}{\alpha}\ln\left(\frac{1}{\alpha}\right) \\ &= \frac{1}{\alpha}\left(\gamma - \ln\alpha + (\alpha - 1)\sum_{k=0}^{\infty} \frac{1}{(k+1)(\alpha k + 1)} \right) \\ &= \delta(\alpha). \end{aligned}$$

This was observed by Dr. Wayne Smith to form the following Corollary:

Corollary 5.

$$\delta(\alpha) = -\frac{1}{\alpha}\frac{\Gamma'\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)} + \frac{1}{\alpha}\ln\left(\frac{1}{\alpha}\right).$$

3. The special case of δ when $\alpha = \frac{1}{n}$.

First, observe that

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(\alpha k+1)} = \sum_{k=0}^{\infty} \frac{\frac{1}{\alpha}}{(k+1)\left(k+\frac{1}{\alpha}\right)}.$$

When we make the substitution, $\alpha = \frac{1}{n}$, we get the result that

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(\alpha k+1)} = \sum_{k=0}^{\infty} \frac{n}{(k+1)(k+n)}.$$

Note that using partial fraction decomposition:

$$\frac{n}{(k+1)(k+n)} = \frac{A}{(k+1)} + \frac{B}{(k+n)}$$

$$n = A(k+n) + B(k+1)$$

$$n = Ak + An + Bk + B$$

$$n = (A+B)k + An + B,$$

which leads to $A = -B$ and $A = \frac{n}{(n-1)}, B = -\frac{n}{(n-1)}$.
So, in general:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{(k+1)(\alpha k+1)} &= \sum_{k=0}^{\infty} \frac{n}{(k+1)(k+n)} \\
&= \frac{n}{(n-1)} \left(\sum_{k=0}^{\infty} \frac{1}{(k+1)} - \sum_{k=0}^{\infty} \frac{1}{(k+n)} \right) \\
&= \frac{n}{(n-1)} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \right).
\end{aligned}$$

Now, substituting $\alpha = \frac{1}{n}$ into to our previously established formula:

$$\begin{aligned}
\delta &= \frac{1}{\alpha} \left(\gamma - \ln \alpha + (\alpha - 1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(\alpha k+1)} \right) \\
&= n \left(\gamma + \ln n + \left(\frac{1}{n} - 1 \right) \left(\frac{n}{n-1} \right) \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \right) \right) \\
&= n \left(\gamma + \ln n + \left(\frac{1-n}{n} \right) \left(\frac{n}{n-1} \right) \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \right) \right) \\
&= n \left(\gamma + \ln n - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \right) \right).
\end{aligned}$$

Example 6. *Let $\alpha = \frac{1}{2}$, then $n = 2$.*

$$\begin{aligned}
\delta\left(\frac{1}{2}\right) &= 2\left(\gamma + \ln(2) - 1\right) \\
&\approx 0.540726.
\end{aligned}$$

Example 7. *Let $\alpha = \frac{1}{5}$, then $n = 5$.*

$$\begin{aligned}\delta\left(\frac{1}{5}\right) &= 5\left(\gamma + \ln(5) - \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right]\right)\\ &\approx 0.516604.\end{aligned}$$

Example 8. *Let $\alpha = \frac{1}{10}$, then $n = 10$.*

$$\begin{aligned}\delta\left(\frac{1}{10}\right) &= 10\left(\gamma + \ln(10) - \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right]\right)\\ &\approx 0.508338.\end{aligned}$$

4. Reduction formulas for integer α:

Suppose we start with the harmonic series

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots .$$

If we then take away every other term we will have

$$\begin{array}{l} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \cdots \\ -\left(+\frac{1}{2} \qquad + \frac{1}{4} \qquad + \frac{1}{6} \qquad + \frac{1}{8} \cdots\right) \\ \hline 1 \qquad + \frac{1}{3} \qquad + \frac{1}{5} \qquad + \frac{1}{7} \qquad \cdots . \end{array}$$

Mathematically,

$$\sum_{k=0}^{n} \frac{1}{2k+1} = \sum_{k=0}^{2n} \frac{1}{k+1} - \frac{1}{2}\sum_{k=0}^{n} \frac{1}{k+1}.$$

Theorem 9. *For $\alpha = 2$,*

$$\delta(2) = \frac{1}{2}(\gamma + \ln(2)).$$

Proof. Using the identities:

$$\begin{aligned}\sum_{k=0}^{n} \frac{1}{2k+1} &= \frac{1}{2}\ln(2n) + \delta(2) + \varepsilon_1, \quad \text{where } \lim_{n\to\infty} \varepsilon_1 = 0,\\ \sum_{k=0}^{2n} \frac{1}{k+1} &= \ln(2n) + \gamma + \varepsilon_2, \quad \text{where } \lim_{n\to\infty} \varepsilon_2 = 0,\end{aligned}$$

and

$$\sum_{k=0}^{n} \frac{1}{k+1} = \ln(n) + \gamma + \varepsilon_3, \quad \text{where } \lim_{n\to\infty} \varepsilon_3 = 0,$$

we see that

$$\sum_{k=0}^{n} \frac{1}{2k+1} = \sum_{k=0}^{2n} \frac{1}{k+1} - \frac{1}{2}\sum_{k=0}^{n} \frac{1}{k+1}$$

can be rewritten as

$$\frac{1}{2}\ln(2n) + \delta(2) = \ln(2n) + \gamma - \frac{1}{2}(\ln(n) + \gamma).$$

It follows that

$$\begin{aligned}\delta(2) &= \ln(2n) + \gamma - \frac{1}{2}\ln(n) - \frac{1}{2}\gamma - \frac{1}{2}\ln(2n) \\ &= \frac{1}{2}\ln(2) + \frac{1}{2}\gamma \\ &= \frac{1}{2}(\gamma + \ln(2)).\end{aligned}$$

□

Continuing with the result of our first difference,

$$\begin{array}{cccccccc} 1+\frac{1}{3} & +\frac{1}{5} & +\frac{1}{7} & +\frac{1}{9} & +\frac{1}{11} & +\frac{1}{13} & +\frac{1}{15}\cdots \\ -\Big(+\frac{1}{3} & & +\frac{1}{7} & & +\frac{1}{11} & & +\frac{1}{15}\cdots\Big) \\ \hline 1 & +\frac{1}{5} & & +\frac{1}{9} & & +\frac{1}{13} & \cdots. \end{array}$$

In other words,

$$\sum_{k=0}^{n}\frac{1}{4k+1} = \sum_{k=0}^{2n}\frac{1}{2k+1} - \frac{1}{3}\sum_{k=0}^{n}\frac{1}{\frac{4}{3}k+1}.$$

This procedure can be generalized by the following:

Theorem 10. *For α even, $\alpha \in \mathbb{Z}^{+}$,*

$$\sum_{k=0}^{n}\frac{1}{\alpha k+1} = \sum_{k=0}^{2n}\frac{1}{\left(\frac{\alpha}{2}\right)k+1} - \frac{1}{\left(\frac{\alpha}{2}+1\right)}\sum_{k=0}^{n}\frac{1}{\left(\frac{\alpha}{\left(\frac{\alpha}{2}+1\right)}\right)k+1}.$$

We have established

$$\begin{array}{cccccccc} 1+\frac{1}{2} & +\frac{1}{3} & +\frac{1}{4} & +\frac{1}{5} & +\frac{1}{6} & +\frac{1}{7} & +\frac{1}{8}\cdots \\ -\Big(+\frac{1}{2} & & +\frac{1}{4} & & +\frac{1}{6} & & +\frac{1}{8}\cdots\Big) \\ \hline 1 & +\frac{1}{3} & & +\frac{1}{5} & & +\frac{1}{7} & \cdots. \end{array}$$

Suppose we now start with the resulting series, instead of the harmonic series, and again subtract every other term,

$$\begin{array}{cccccccc} 1+\frac{1}{3} & +\frac{1}{5} & +\frac{1}{7} & +\frac{1}{9} & +\frac{1}{11} & +\frac{1}{13} & +\frac{1}{15}\cdots \\ -\Big(+\frac{1}{3} & & +\frac{1}{7} & & +\frac{1}{11} & & +\frac{1}{15}\cdots\Big) \\ \hline 1 & +\frac{1}{5} & & +\frac{1}{9} & & +\frac{1}{13} & \cdots. \end{array}$$

So we have started with

$$\sum_{k=0}^{2n}\frac{1}{2k+1},$$

and subtracted

$$\frac{1}{3}\sum_{k=0}^{n}\frac{1}{\frac{4}{3}k+1},$$

to get

$$\sum_{k=0}^{n}\frac{1}{4k+1} = \sum_{k=0}^{n}\frac{1}{\left(2^{2}\right)k+1}.$$

In general,

$$\sum_{k=0}^{n}\frac{1}{4k+1}=\sum_{k=0}^{2n}\frac{1}{2k+1}-\frac{1}{3}\sum_{k=0}^{n}\frac{1}{\frac{4}{3}k+1},$$

but

$$\sum_{k=0}^{2n}\frac{1}{2k+1}=\sum_{k=0}^{4n}\frac{1}{k+1}-\frac{1}{2}\sum_{k=0}^{2n}\frac{1}{k+1},$$

so

$$\sum_{k=0}^{n}\frac{1}{4k+1}=\sum_{k=0}^{4n}\frac{1}{k+1}-\frac{1}{2}\sum_{k=0}^{2n}\frac{1}{k+1}-\frac{1}{3}\sum_{k=0}^{n}\frac{1}{\frac{4}{3}k+1}.$$

For exponential increments of 2, we may generalize thusly:

Theorem 11. *For $\alpha = 2^m$, where $\alpha, m \in \mathbb{Z}^+$,*

$$\sum_{k=0}^{n}\frac{1}{2^m k+1}=\sum_{k=0}^{2^m n}\frac{1}{k+1}-\left[\sum_{n=1}^{m}\frac{1}{2^{n-1}+1}\left(\sum_{k=0}^{n-1}\frac{1}{\frac{2^n}{2^{n-1}+1}k+1}\right)\right].$$

But what about when alpha is not necessarily an even number? We simply have to remove more terms. For example:

$$\begin{array}{l}
1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\cdots \\
-\left(+\frac{1}{2}\qquad\qquad +\frac{1}{5}\qquad\qquad +\frac{1}{8}\qquad\qquad \cdots\right) \\
-\left(\qquad +\frac{1}{3}\qquad\qquad +\frac{1}{6}\qquad\qquad +\frac{1}{9}\qquad\qquad \cdots\right) \\
\hline
1\qquad\qquad +\frac{1}{4}\qquad\qquad +\frac{1}{7}\qquad\qquad +\frac{1}{10}\qquad \cdots.
\end{array}$$

Using mathematical notation,

$$\sum_{k=0}^{n}\frac{1}{3k+1}=\sum_{k=0}^{3n}\frac{1}{k+1}-\frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{\frac{3}{2}k+1}-\frac{1}{3}\sum_{k=0}^{n-1}\frac{1}{k+1}.$$

Generally,

Theorem 12. *For positive integer α and integer, $q > 1$:*

$$\sum_{k=0}^{n}\frac{1}{\alpha k+1}=\sum_{k=0}^{\alpha n}\frac{1}{k+1}-\sum_{q=2}^{\alpha}\frac{1}{q}\left[\sum_{k=0}^{n-1}\frac{1}{\frac{\alpha}{q}k+1}\right].$$

For another interesting reduction formula, let

$$\frac{1}{1}+\frac{1}{\alpha+1}+\frac{1}{2\alpha+1}+\cdots+\frac{1}{\alpha n+1}=\frac{1}{1}+\frac{1}{\alpha+1}\left(\frac{1}{1}+\frac{\alpha+1}{(2\alpha+1)}+\frac{\alpha+1}{(3\alpha+1)}+\cdots+\frac{\alpha+1}{(\alpha(n-1)+1)}\right).$$

The left hand side of the equation is:

$$\frac{1}{\alpha}\ln(\alpha)+\frac{1}{\alpha}\ln(n)+\delta(\alpha)+\varepsilon_1=\frac{1}{1}+\frac{1}{\alpha+1}\left(\frac{1}{1}+\frac{1}{b+1}+\frac{1}{2b+1}+\cdots+\frac{1}{(n-1)b+1}\right), \tag{3}$$

where $\lim_{n\to\infty}\varepsilon_1 = 0$.

Now,

$$\begin{aligned}(\alpha+1)(b+1) &= 2\alpha+1,\\ \alpha b+\alpha+b+1 &= 2\alpha+1,\\ \alpha b+b &= \alpha,\\ b &= \frac{\alpha}{\alpha+1}.\end{aligned}$$

So, from (3),

$$\frac{1}{\alpha}\ln(\alpha)+\frac{1}{\alpha}\ln(n)+\delta(\alpha)+\varepsilon_1=\frac{1}{1}+\frac{1}{\alpha+1}\left(\frac{1}{b}\ln(b)+\frac{1}{b}\ln(n)+\delta(b)+\varepsilon_2\right),\text{where }\lim_{n\to\infty}\varepsilon_2=0;$$

substituting $b=\frac{\alpha}{\alpha+1}$:

$$\frac{1}{\alpha}\ln(\alpha)+\frac{1}{\alpha}\ln(n)+\delta(\alpha)+\varepsilon_1=1+\frac{1}{\alpha}\ln(\alpha)+\frac{1}{\alpha}\ln(n)-\frac{1}{\alpha}\ln(\alpha+1)+\frac{1}{\alpha+1}\delta\left(\frac{\alpha}{\alpha+1}\right)+\varepsilon_2.$$

Simplifying, we get the following theorem:

Theorem 13.

$$\delta(\alpha)=1-\frac{1}{\alpha}\ln(\alpha+1)+\left(\frac{1}{\alpha+1}\right)\delta\left(\frac{\alpha}{\alpha+1}\right).$$

Example 14. *Let* $\alpha=1$, *then*

$$\delta(1)=\gamma=1-\ln(2)+\frac{1}{2}\cdot\delta\left(\frac{1}{2}\right).$$

Example 15. *Let* $\alpha=2$, *then*

$$\delta(2)=1-\frac{1}{2}\ln(3)+\frac{1}{3}\cdot\delta\left(\frac{2}{3}\right).$$

Example 16. *Let* $\alpha=\frac{2}{3}$, *then*

$$\delta\left(\frac{2}{3}\right)=1-\frac{3}{2}\ln\left(\frac{5}{3}\right)+\frac{5}{3}\cdot\delta\left(\frac{2}{5}\right).$$

Example 17. *Let* $\alpha=3$, *then*

$$\delta(3)=1-\frac{1}{3}\ln(4)+\frac{1}{4}\cdot\delta\left(\frac{3}{4}\right).$$

5. Applying the Euler-MacLaurin summation formula.

The Euler-MacLaurin summation formula compares $\sum_{k=0}^{n} f(k)$ to $\int_0^n f(x)dx$, for a function $f(x)$ in $C^{2m+1}([0,n])$.

$$\sum_{k=0}^{n}f(k)=\int_0^n f(x)dx+\frac{1}{2}(f(0)+f(n))+\sum_{k=1}^{m}\frac{B_{2k}}{(2k)!}(f^{2k-1}(n)-f^{2k-1}(0))+R_n(f,m),$$

where

$$R_n(f,m)\le\frac{2}{(2\pi)^{2m}}\int_0^n\left|f^{2m+1}(x)\right|dx.$$

Here the B_j denote the Bernoulli Numbers,

$$
\begin{aligned}
B_0 &= 1 \\
B_1 &= \frac{1}{2} \\
B_2 &= \frac{1}{6} \\
B_3 &= 0 \\
B_4 &= -\frac{1}{30} \\
B_5 &= 0 \\
B_6 &= \frac{1}{42} \\
B_7 &= 0 \\
B_8 &= -\frac{1}{30} \\
&\cdots
\end{aligned}
$$

and f^j denotes the j^{th} derivative of f.

Substituting the B_j's into the formula, yields

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)dx + \frac{1}{2}(f(0)+f(n)) + \frac{\left(\frac{1}{6}\right)}{2!}(f'(n)-f'(0)) + \frac{\left(-\frac{1}{30}\right)}{4!}(f'''(n)-f'''(0)) + \frac{\left(\frac{1}{42}\right)}{6!}(f'''''(n)-f'''''(0)) + \cdots$$

Now for the function $f(x) = \frac{1}{\alpha x+1}$, we have:

$$
\begin{aligned}
\int_0^n f(x)dx &= \frac{1}{\alpha}\ln(\alpha x+1), \\
f'(x) &= -\frac{\alpha}{(\alpha x+1)^2}, \\
f''(x) &= \frac{2\alpha^2}{(\alpha x+1)^3}, \\
f'''(x) &= -\frac{6\alpha^3}{(\alpha x+1)^4}, \\
f^{(4)}(x) &= \frac{24\alpha^4}{(\alpha x+1)^5}, \\
f^{(5)}(x) &= -\frac{120\alpha^5}{(\alpha x+1)^6}, \\
&\cdots
\end{aligned}
$$

and in general,

$$f^{(k)}(x) = \frac{(-1)^k k! \alpha^k}{(\alpha x+1)^{k+1}}.$$

Thus, we obtain

$$
\begin{aligned}
\sum_{k=0}^{n} \frac{1}{\alpha k+1} = \int_0^n \frac{1}{\alpha x+1}dx + \frac{1}{2}\left(1 + \frac{1}{\alpha n+1}\right) + \frac{B_2}{2!}\left(-\frac{\alpha}{(\alpha n+1)^2} + \alpha\right) + \\
\frac{B_4}{4!}\left(-\frac{6\alpha^3}{(\alpha n+1)^4} + 6\alpha^3\right) + \frac{B_6}{6!}\left(-\frac{120\alpha^5}{(\alpha n+1)^6} + 120\alpha^5\right) + \cdots + R_m
\end{aligned}
$$

$$= \frac{1}{\alpha}\ln(\alpha x+1) + \frac{1}{2}\left(1+\frac{1}{\alpha n+1}\right) + \frac{\alpha}{12}\left(\frac{-1}{(\alpha n+1)^2}+1\right) - \frac{\alpha^3}{120}\left(\frac{-1}{(\alpha n+1)^4}+1\right) - \frac{\alpha^5}{252}\left(\frac{-1}{(\alpha n+1)^6}+1\right) + \cdots + \frac{-B_{2m}}{2m}\alpha^{2m-1}\left(\frac{-1}{(\alpha n+1)^{2m}}+1\right) + R_m,$$

whence

$$\begin{aligned}
\delta(\alpha) &= \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{1}{\alpha k+1} - \int_0^n \frac{1}{\alpha x+1}dx\right)\\
&= \frac{1}{2} + \frac{\alpha}{12} - \frac{\alpha^3}{120} + \frac{\alpha^5}{252} - \frac{\alpha^7}{240}\cdots + \frac{B_{2m}}{2m}\alpha^{2m-1} + R_m,
\end{aligned}$$

where

$$\begin{aligned}
R_m &\le \frac{2}{(2\pi)^{2m}}\int_0^n \left|f^{2m+1}\right| dx = \frac{2}{(2\pi)^{2m}}\left(-f^{2m}\right)\Big|_0^n\\
&= -\frac{2}{(2\pi)^{2m}}\cdot\frac{(2m)!\alpha^{2m}}{(\alpha x+1)^{2m+1}}\Big|_0^n\\
&= \frac{2\,(2m)!\alpha^{2m}}{(2\pi)^{2m}} - \frac{2}{(2\pi)^{2m}}\cdot\frac{(2m)!\alpha^{2m}}{(\alpha n+1)^{2m+1}}\\
&\le \frac{2\,(2m)!\alpha^{2m}}{(2\pi)^{2m}}\\
&= C_{2m}\alpha^{2m}
\end{aligned}$$

where the constants C_{2m} are given by the table below. This series:

$$\frac{1}{2} + \frac{\alpha}{12} - \frac{\alpha^3}{120} + \frac{\alpha^5}{252} - \frac{\alpha^7}{240}\cdots + \frac{B_{2k}}{2k}\alpha^{2k-1}\cdots,$$

does not converge because the Bernoulli Numbers grow rapidly after the first few terms. Asypmtotically,

$$B_{2m} \approx (-1)^{n-1}\,4\sqrt{\pi m}\left(\frac{m}{\pi e}\right)^{2m}.$$ [8]

Following a suggestion of Hardy[6], we take the first few terms only and estimate the remainder term, which will be small if $\alpha < 1$.

$$R_{2m} \le \frac{2\,(2m)!\alpha^{2m}}{(2\pi)^{2m}} = C_{2m}\alpha^{2m}.$$

The coefficients, C_k, are in the following table:

k	$2\cdot k!$	$(2\pi)^k$	$\frac{2\cdot k!}{(2\pi)^k}$
2	2	39.4	.101
4	24	1,555	.0309
6	720	61,342	.0235
8	40,320	2.4×10^6	.0333
10	3,628,800	9.5×10^7	.076
12	479,001,600	3.7×10^9	.25

Also,

$$\delta(\alpha) = \frac{1}{2} + \frac{\alpha}{12} + \mathcal{O}\left(\alpha^2\right)$$

shows

$$\lim_{\alpha\to0}\delta(\alpha) = \frac{1}{2}$$

and

$$\lim_{\alpha \to 0} \delta'(\alpha) = \frac{1}{12}.$$

Example 18. *Suppose* $\alpha = \frac{2}{11}$ *and* $2m = 6$. *Then,*

$$\delta\left(\frac{2}{11}\right) = \frac{1}{2} + \frac{\left(\frac{2}{11}\right)}{12} - \frac{\left(\frac{2}{11}\right)^3}{120} + \frac{\left(\frac{2}{11}\right)^5}{252} - \cdots - \frac{B_{2k}}{2k}\left(\frac{2}{11}\right)^{2k-1} + \cdots.$$

Even evaluating the first few terms we see

$$\delta\left(\frac{2}{11}\right) \approx \frac{1}{2} + \frac{\left(\frac{2}{11}\right)}{12} - \frac{\left(\frac{2}{11}\right)^3}{120} + \frac{\left(\frac{2}{11}\right)^5}{252} = \frac{26\,131\,604}{50\,731\,065} \approx 0.5151006$$

and the error is given as

$$R_6 \leq .0235\left(\frac{2}{11}\right)^6 = 8.489\,7 \times 10^{-7}.$$

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